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QUADRATIC INTEGRALS AND THE REDUCIBILITY OF THE EQUATIONS OF MOTION OF A COMPLEX MECHANICAL SYSTEM IN A CENTRAL FIELD†

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A mechanical system, consisting of a non-variable rigid body (a carrier) and a subsystem, the configuration and composition of which may vary with time (the motion of its elements with respect to the carrier is specified), is considered. The system moves in a central force field at a distance from its centre which considerably exceeds the dimensions of the system. The effect of the system motion about the centre of mass on the motion of the centre of mass, which is assumed to be known, is ignored (the analogue of the limited problem [1] for a rigid body). The necessary and sufficient conditions for a quadratic integral of the motion around the centre of mass to exist are obtained in the case when there is no dynamic symmetry. It is shown that, for a quadratic integral to exist, it is necessary that the trajectory of the motion of the centre of mass should be on the surface of a certain circular cone, fixed in inertial space, with its vertex at the centre of the force field. If the trajectory does not lie on the generatrix of the cone, only one non-trivial quadratic integral can exist and the initial system, in the presence of this quadratic integral, reduces to autonomous form. For the motion of the centre of mass along the generatrix or the motion of the system around a fixed centre of mass, the necessary and sufficient conditions for a non-trivial quadratic integral to exist are obtained, which are generalizations of the energy integral, the de Brun integral [2] and the integral of the projection of the kinetic moment. When three non-trivial quadratic integrals exist, the condition for reduction to an autonomous system describing the rotation of the rigid body around the centre of mass and integrable in quadratures are indicated [3, 4].

1. THE EQUATIONS OF MOTION

Suppose E_0 is an inertial frame of reference with origin at the centre O of a force field, E_1 is an orbital frame of reference with origin at the centre of mass **C** of a mechanical system, E_2 is a frame of reference connected with the carrier, and E_3 is the principal frame of reference with origin at the point **C** and axes coinciding with the principal central axes of inertia of the system.

We will choose an orthobasis $\{\mathbf{g}_i\}$ of the frame of reference E_1 as follows: $\mathbf{g}_3 = \mathbf{r}^0$, $\mathbf{r} = \mathbf{OC}$, $\mathbf{g}_2 = (\mathbf{r} \times (\mathbf{r})_0^\bullet)^0$, $\mathbf{g}_1 = \mathbf{g}_2 \times \mathbf{g}_3$. Here $()_i^\bullet$ is the derivative with respect to time in frame of reference E_i and $\mathbf{a}^0$ is the unit vector of the vector **a**. We can obtain the following formulae for the derivatives of the unit vectors $\mathbf{g}_i$ in frame of reference E_0

$$(\mathbf{g}_i)_0^\bullet = \mathbf{\Omega} \times \mathbf{g}_i, \quad i = 1,2,3, \quad \mathbf{\Omega} = k_2\mathbf{g}_2 + k_3\mathbf{g}_3 \tag{1.1}$$

Here

$$k_2 = |\mathbf{r} \times (\mathbf{r})_0^\bullet| r^{-2}, \quad k_3 = \langle \mathbf{r}, (\mathbf{r})_0^\bullet, (\mathbf{r})_0^{\bullet\bullet} \rangle |\mathbf{r} \times (\mathbf{r})_0^\bullet|^{-2} r \tag{1.2}$$

Putting $\boldsymbol{\gamma} = \mathbf{g}_3$ and denoting the velocity of the centre of the mass by v, we can also write

$$k_2 = v\, |\boldsymbol{\gamma} \times d\boldsymbol{\gamma}/ds|, \quad k_3 = v\, |\boldsymbol{\gamma} \times d\boldsymbol{\gamma}/ds|^{-2} \langle \boldsymbol{\gamma},\ d\boldsymbol{\gamma}/ds,\ d^2\boldsymbol{\gamma}/ds^2 \rangle \tag{1.3}$$

In this limited problem the function $\mathbf{r}(t)$ is specified in frame of reference E_0, and consequently, the functions $k_2(t)$ and $k_3(t)$ are also specified. The position of the frame of reference E_1 with respect to E_0 is also known at each instant of time.

The equations of rotational motion of the carrier of a system of variable composition are known [5]. Another form of these equations, proposed in [6, 7], has the form

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$$(\mathbf{y})_2^{\bullet} = \mathbf{y} \times \mathbf{x}_{20} + \Lambda \mathbf{x}_{20} + \mathbf{L}, \quad \mathbf{y} = J\mathbf{x}_{20} + \mathbf{G}_2 \tag{1.4}$$

Here J is the inertia operator of the system at its centre of mass. $\mathbf{x}_{ij}$ is the angular velocity of the frame of reference E_i with respect to E_j, $\mathbf{L} = \mathbf{M} + \mathbf{M}_2^f + (\mathbf{G}_2)_2^{\bullet}$, $\mathbf{M} = \mathbf{M}^e + \mathbf{M}_r + \mathbf{M}^*$, $\mathbf{M}^e$ and $\mathbf{M}_r$ are the principal moments (with respect to C) of the external and reactive forces, $\mathbf{M}^*$ is a certain control moment, specified in E_2, and

$$\mathbf{G}_i = \Sigma m_n \mathbf{r}_n \times (\mathbf{r}_n)_i^{\bullet}, \quad \mathbf{M}_i^f = -\Sigma m_n \mathbf{r}_n \times (\mathbf{r}_n)_i^{\bullet\bullet} \tag{1.5}$$

The summation is carried out over all elements of the system, m_n is the mass of the point mass M_n and $\mathbf{r}_n = \mathbf{CM}_n$. The symmetrical operator Λ is given by the identity

$$\Lambda \mathbf{x} \equiv J^{\bullet}\mathbf{x} - \Sigma m_n [\mathbf{r}_n^{\bullet} \times (\mathbf{x} \times \mathbf{r}_n) + \mathbf{r}_n \times (\mathbf{x} \times \mathbf{r}_n^{\bullet})]$$

The following equalities hold [7]

$$J\mathbf{x}_{ij} = \mathbf{G}_j - \mathbf{G}_i, \quad \Lambda \mathbf{x}_{ij} = \mathbf{M}_j^f - \mathbf{M}_i^f + (\mathbf{G}_j)_j^{\bullet} - (\mathbf{G}_i)_i^{\bullet} \tag{1.6}$$

When the dimensions of the system are much less than the distance from its centre of mass to the centre O of the field with potential $U(r)$, the principal moment of the external forces is

$$\mathbf{M}^e = p\boldsymbol{\gamma} \times J\boldsymbol{\gamma}, \quad p = -r(r^{-1}U'(r))' \tag{1.7}$$

System (1.4), in this case, taking into account properties (1.6), can be written in the form

$$\mathbf{y}^{\bullet} = \mathbf{y} \times \mathbf{x} + \Lambda \mathbf{x} + p\boldsymbol{\gamma} \times J\boldsymbol{\gamma} + \mathbf{N}, \quad \mathbf{y} = J\mathbf{x} + \mathbf{K} \tag{1.8}$$

$$()^{\bullet} = ()_3^{\bullet}, \quad \mathbf{x} = \mathbf{x}_{30}, \quad \mathbf{K} = \mathbf{G}_3, \quad \mathbf{N} = \mathbf{M}_3^f + \mathbf{K}^{\bullet} + \mathbf{M}_r + \mathbf{M}^*$$

For a Newtonian gravity field $p = 3\mu r^{-3}$.
Taking formulae (1.1) into account, we can supplement the system with Poisson's equations

$$\mathbf{g}_i^{\bullet} = \mathbf{g}_i \times (\mathbf{x} - \boldsymbol{\Omega}), \quad i = 1,2,3 \tag{1.9}$$

System of equations (1.8) and (1.9), when there is no dynamic symmetry, will be called the fundamental dynamical system.

2. BASIC RESULTS

Equations (1.9) have six trivial integrals

$$g_i^2 = \text{const}, \quad (\mathbf{g}_i, \mathbf{g}_j) = \text{const}$$

We know [1], that in the case of the motion of a gyrostat in a Newtonian field in a circular orbit the fundamental dynamical system has a quadratic integral (integral (2.2) with constants r, k_2, $p = 3\mu r^{-3}$ and $k_3 \equiv 0$), while the motion of the rigid body around a fixed centre of mass possesses three non-trivial quadratic integrals: the energy integral, the integral of the projection of the kinetic moment and the de Brun integral [2] (integrals (2.28)–(2.30) with $p = 3\mu r^{-3}$ and $r = \text{const}$).
The arbitrary quadratic integral of the fundamental dynamical system can be written in the form

$$(\mathbf{y}, F\mathbf{y}) + \sum_{i=1}^{3} [(\mathbf{g}_i, P_i\mathbf{g}_i) + (Q_i\mathbf{y}, \mathbf{g}_i) + (\mathbf{n}_i, \mathbf{g}_i)] + (\mathbf{m}, \mathbf{y}) + h(t) +$$
$$+ (R_1\mathbf{g}_2, \mathbf{g}_3) + (R_2\mathbf{g}_3, \mathbf{g}_1) + (R_3\mathbf{g}_1, \mathbf{g}_2) = \text{const} \tag{2.1}$$

Below we will solve the problem of obtaining the conditions for non-trivial quadratic integrals to exist and we will obtain in explicit form the symmetrical operators F, P_i, the operators Q_i, R_i, and the parameters $\mathbf{n}_i$, $\mathbf{m}$ and h. We will assume that the operators and parameters listed here are differentiable functions of time.

We will consider separately the case when $k_2 \equiv 0$, corresponding to the centre of mass at rest in the frame of reference E_0, or its motion along a straight line, fixed in E_0, passing through the centre of the field, and the case when $k_2 \neq 0$.

Theorem 1. For a non-trivial quadratic integral of the fundamental dynamical system to exist in the case when $k_2 \neq 0$ it is necessary and sufficient for the following conditions to be satisfied

1) the centre of mass of the system moves along the surface of an arbitrary circular cone, fixed in inertial space, with vertex at the centre of the force field;

2) the velocity υ of the centre of mass varies as follows:

$$\upsilon = c_2 |p|^{1/2} r^2 |\mathbf{r} \times d\mathbf{r}/ds|^{-1}, \quad c_2 = \text{const}$$

3) the inertia operator of the system varies similarly, $J = \eta J_0$;

4) $\Lambda = (\ln \zeta)^{\cdot} J$, $\zeta = k_2 \eta$;

5) $\mathbf{N} = \mathbf{K}^{\cdot}$;

6) $\mathbf{K} = \zeta \mathbf{K}_0$.

Here J_0 and $\mathbf{K}_0$ are certain constant operators and vectors in the principal frame of reference. The integral in this case can be written in the form

$$p^{-1}[(\mathbf{x}, J_0\mathbf{x}) + p(\boldsymbol{\gamma}, J_0\boldsymbol{\gamma}) - 2(J_0\mathbf{x} + k_2\mathbf{K}_0, \boldsymbol{\Omega})] = \text{const} \tag{2.2}$$

or, in equivalent form,

$$p^{-1}[(\mathbf{x}_{31}, J_0\mathbf{x}_{31}) + p(\boldsymbol{\gamma}, J_0\boldsymbol{\gamma}) - (\boldsymbol{\Omega}, J_0\boldsymbol{\Omega}) - 2k_2(\mathbf{K}_0, \boldsymbol{\Omega})] = \text{const} \tag{2.3}$$

Condition 1 is satisfied, in particular, for any trajectory in a plane containing the centre of the force field.

Note that, when the centre of mass C of the system moves along the surface of the cone indicated in Condition 1, the orbital frame of reference E_1 chosen above has the following orientation: the unit vector $\mathbf{g}_1$ is directed along the tangent to the circular section of the cone, passing through the point C, the unit vector $\mathbf{g}_2$ is directed along the normal to the surface of the cone and the unit vector $\mathbf{g}_3$ is directed along the generatrix of the cone.

For free motion of a point mass in a central field in a Kepler orbit with velocity υ_k we have $\upsilon_k|\mathbf{r} \times d\mathbf{r}/ds| = \text{const}$, and Condition 2 for the velocity υ of the controlled motion of the centre of mass can be written in the form

$$\upsilon = \text{const}\, |p|^{1/2} r^2 \upsilon_k$$

When the conditions of Theorem 1 are satisfied, the system reduces to the following autonomous form

$$J_0 d\mathbf{u}/d\tau = (J_0\mathbf{u} + \mathbf{K}_0) \times \mathbf{u} + c_3\mathbf{g}_3 \times J_0\mathbf{g}_3 \tag{2.4}$$

$$d\mathbf{g}_i/d\tau = \mathbf{g}_i \times (\mathbf{u} - \mathbf{g}_2 - c_1\mathbf{g}_3), \quad i = 1, 2, 3 \tag{2.5}$$

Here the differentiation is in the frame of reference E_3, $c_3 = (c_2)^{-2}$ and

$$\mathbf{u} = \mathbf{x}/k_2, \quad d\tau = k_2 dt = |\mathbf{r}^0 \times d\mathbf{r}^0| \tag{2.6}$$

It can be checked directly that the integral of system (2.4), (2.5) is the following integral, which is identical with integral (2.2) (Proposition 32)

$$(\mathbf{u}, J_0\mathbf{u}) + c_3(\mathbf{g}_3, J_0\mathbf{g}_3) - 2(J_0\mathbf{u} + \mathbf{K}_0, \mathbf{g}_2 + c_1\mathbf{g}_3) = \text{const} \tag{2.7}$$

It follows from Theorem 1 that, in the special case of the motion of a system of constant composition ($\Lambda \equiv 0$) in a Newtonian field ($p = 3\mu r^{-3}$), for a quadratic integral to exist when $k_2 \neq 0$ it is necessary and sufficient for Condition 1 of Theorem 1 to be satisfied and also the conditions

$$J = k_2^{-1} J_0, \quad \mathbf{N} = 0, \quad \mathbf{K} = \mathbf{K}_0, \quad \upsilon = c_2 r^{1/2} |\mathbf{r} \times d\mathbf{r}/ds|^{-1}$$

Theorem 2. If, when $k_2 \equiv 0$, a non-trivial quadratic integral of the fundamental dynamical system exists, it can be represented in the form

$$(J\mathbf{x}, FJ\mathbf{x}) + (\boldsymbol{\gamma}, P_3\boldsymbol{\gamma}) + \nu(J\mathbf{x} + \mathbf{K}, \boldsymbol{\gamma}) = \text{const} \tag{2.8}$$

The eigenvalues φ_i and p_{3i} of the operators F and P_3 have the form

$$\varphi_i = (pa_i)^{-1}\sigma_i, \quad a_i = A_i\Delta A_i \tag{2.9}$$

$$p_{3i} = -\tfrac{1}{3}\Delta\sigma_i \tag{2.10}$$

and the constants σ_i are related by the condition

$$\sigma_1 + \sigma_2 + \sigma_3 = 0 \tag{2.11}$$

For the integral to exist it is necessary and sufficient for the following conditions to be satisfied

$$\nu\Lambda + \nu^{\bullet}J = 0 \tag{2.12}$$

$$\nu\mathbf{N} + \nu^{\bullet}\mathbf{K} = 0 \tag{2.13}$$

$$\lambda_{ij}(\sigma_i\Delta A_j + \sigma_j\Delta A_i) = K^{(k)}(\sigma_i\Delta A_j - \sigma_j\Delta A_i)\delta_{ijk} \quad (i,j,k) \tag{2.14}$$

$$\sigma_i(\mathbf{N} - \mathbf{K}^{\bullet})^{(i)} = 0, \quad i = 1,2,3 \tag{2.15}$$

$$\sigma_i[(\ln pa_i)^{\bullet} - 2\lambda_{ii}A_i^{-1}] = 0, \quad i = 1,2,3 \tag{2.16}$$

Here and henceforth

$$\Delta A_i = (A_j - A_k)\delta_{ijk}, \quad (\mathbf{m})^{(i)} = (\mathbf{m}, \mathbf{e}_i)$$

Note that, for a quadratic integral in complete form (all $\sigma_i \neq 0$), condition (2.15) takes the same form as in the problem of the motion of a system around a fixed point of a carrier in a uniform gravitational field [8] and in the problem of free motion [7], $\mathbf{N} = \mathbf{K}^{\bullet}$. This condition can be written in the form

$$\mathbf{M}_3^f + \mathbf{M}_r + \mathbf{M}^{\bullet} = 0$$

Below we will show (Proposition 21), that integral (2.8) can be written in the form

$$\begin{aligned}&\psi_1[\mathbf{x}, J\mathbf{x}) + p(\boldsymbol{\gamma}, J\boldsymbol{\gamma})] + \psi_2[(J\mathbf{x})^2 - pA_1A_2A_3(\boldsymbol{\gamma}, J^{-1}\boldsymbol{\gamma})] + \\ &+ \nu(J\mathbf{x} + \mathbf{K}, \boldsymbol{\gamma}) = \text{const}\end{aligned} \tag{2.17}$$

Under certain conditions this integral can be expanded in three non-trivial quadratic integrals, which are generalizations of the energy integral, the de Brun integral and the integral of the projection of the kinetic moment.

When the integral of the projection of the kinetic moment exists, the system reduces to the form (Proposition 28)

$$\mathbf{v}^{\bullet} = \mathbf{v} \times \mathbf{x} + p\boldsymbol{\gamma} \times J'\boldsymbol{\gamma}, \quad \boldsymbol{\gamma}^{\bullet} = \boldsymbol{\gamma} \times \mathbf{x}, \quad \mathbf{v} = J'\mathbf{x} + \mathbf{K}' \tag{2.18}$$

Theorem 3. If, when $k_2 \equiv 0$, three independent non-trivial quadratic integrals of the fundamental dynamical system exist, they can be written in the form

$$\nu^2(J\mathbf{x}, J_0^{-1}J\mathbf{x}) + (\boldsymbol{\gamma}, J_0\boldsymbol{\gamma}) = \text{const} \tag{2.19}$$

$$(\nu J\mathbf{x})^2 - A_{10}A_{20}A_{30}(\boldsymbol{\gamma}, J_0^{-1}\boldsymbol{\gamma}) = \text{const} \tag{2.20}$$

$$\nu(J\mathbf{x}, \boldsymbol{\gamma}) = \text{const} \tag{2.21}$$

For these integrals to exist it is necessary and sufficient for the following conditions to be satisfied

$$\Lambda = -(\ln \nu)^{\bullet} J, \quad \mathbf{K} = 0, \quad \mathbf{N} = 0, \quad p\nu^2 a_i = a_{i0}, \quad i = 1, 2, 3 \tag{2.22}$$

When the conditions of this theorem are satisfied or, which is the same thing, when there are three non-trivial quadratic integrals, the system can be reduced to the form

$$J_0 \frac{d\mathbf{w}}{d\tau} = (J_0\mathbf{w}) \times \mathbf{w} + \lambda J^{-1} J_0(\boldsymbol{\gamma} \times J_0\boldsymbol{\gamma}), \quad \frac{d\boldsymbol{\gamma}}{d\tau} = \lambda\boldsymbol{\gamma} \times J^{-1} J_0\mathbf{w} \tag{2.23}$$

where the differentiation is carried out in the frame of reference E_3 and

$$\mathbf{w} = \nu J_0^{-1} J\mathbf{x}, \quad d\tau = \frac{dt}{\lambda\nu}, \quad \lambda = p\nu^2 \frac{A_1A_2A_3}{A_{10}A_{20}A_{30}} \tag{2.24}$$

When $J = \lambda J_0$, we obtain the well-known autonomous system, integrable in quadratures [3, 4], which describes the rotation of a rigid body around a fixed centre of mass.

Theorem 4. When $k_2 \equiv 0$ and the conditions $\mathbf{K} = 0$, $\mathbf{N} = 0$ are satisfied as well as the conditions

$$J = (p\nu^2)^{-1/2} J_0, \quad \Lambda = -(p\nu^2)^{-1/2}(\ln \nu)^{\bullet} J_0 \tag{2.25}$$

the fundamental dynamical system can be reduced to the autonomous system

$$J_0 \frac{d\mathbf{z}}{d\tau} = (J_0\mathbf{z}) \times \mathbf{z} + \boldsymbol{\gamma} \times J_0\boldsymbol{\gamma}, \quad \frac{d\boldsymbol{\gamma}}{d\tau} = \boldsymbol{\gamma} \times \mathbf{z} \tag{2.26}$$

$$\mathbf{z} = p^{-1/2}\mathbf{x}, \quad d\tau = p^{1/2} dt \tag{2.27}$$

When the conditions of Theorem 4 are satisfied, integrals (2.19)–(2.21) can be written in the form

$$p^{-1}(J_0\mathbf{x}, \mathbf{x}) + (\boldsymbol{\gamma}, J_0\boldsymbol{\gamma}) = \text{const} \tag{2.28}$$

$$p^{-1}(J_0\mathbf{x})^2 - A_{10}A_{20}A_{30}(\boldsymbol{\gamma}, J_0^{-1}\boldsymbol{\gamma}) = \text{const} \tag{2.29}$$

$$p^{-1/2}(J_0\mathbf{x}, \boldsymbol{\gamma}) = \text{const} \tag{2.30}$$

3. PROOF OF THEOREM 1

Proposition 1. The operators P_1 and P_2 are proportional to the identity operator E.

Proof. Differentiating integral (2.1), by virtue of the fundamental dynamical system we obtain an identity which we will call the fundamental identity (we will not write it here due to its length). Separating out, in this identity, terms with g_1x, we obtain the identity $\langle P_1\mathbf{g}_1, \mathbf{g}_1, \mathbf{x}\rangle \equiv 0$, which is only satisfied when $P_1 = \varphi(t)E$. Equating the terms with g_2x to zero we obtain the condition $P_2 = \upsilon(\overset{2}{t})E$.

Proposition 2. The operators R_i ($i = 1, 2, 3,$) are proportional to the identity operator.

Proof. Separating out, in the fundamental identity, the terms with g_1g_2x (taking into account the fact that $\mathbf{g}_1 \times \mathbf{g}_2 = \mathbf{g}_3$), we obtain the following identity (B^* is the operator conjugate to B)

$$(R_3^*\mathbf{g}_2) \times \mathbf{g}_1 + (R_3\mathbf{g}_1) \times \mathbf{g}_2 \equiv 0$$

This identity is only satisfied if $R_3 = kE$. In the same way we can obtain similar representations for the operators R_1 and R_2.

Proposition 3. The operators Q_i have the form

$$Q_1 = 0, \quad Q_i = \nu_i E, \quad i = 2, 3$$

Proof. The following identity is obtained from the fundamental identity by separating out terms with $x^2 g_1$

$$\langle Q_1^* \mathbf{g}_1, J\mathbf{x}, \mathbf{x}\rangle + \langle Q_1 J\mathbf{x}, \mathbf{g}_1, \mathbf{x}\rangle \equiv 0$$

that is equivalent to the identity

$$Q_1(J\mathbf{x} \times \mathbf{x}) + \mathbf{x} \times Q_1 J\mathbf{x} \equiv 0$$

which is only satisfied if $Q_1 = \nu_1 E$. Equating terms with $x^2 g_2$ and $x^2 g_3$ to zero in the fundamental identity we obtain the conditions $Q_i = \nu_i E$ $(i = 2, 3)$. If we now separate out, in the fundamental identity, the terms with $g_2 g_3$, we obtain the identity

$$p\nu_1 \mathbf{g}_2 \times J\mathbf{g}_3 \equiv 0$$

which, when $p \neq 0$, is only satisfied if $\nu_1 \equiv 0$.

Note that when there are trivial integrals, the terms $(\mathbf{g}_1, P_1\mathbf{g}_1)$, $(\mathbf{g}_2, P_2\mathbf{g}_2)$ in integral (2.1), by virtue of Proposition 1, are equal to $\varphi(t)$ and $\nu(t)$ respectively, and they can be included in the term $h(t)$. Terms of the form $(R_i\mathbf{g}_j, \mathbf{g}_k)$ are equal to zero by virtue of Proposition 2.

Integral (2.1) and the fundamental identity can now be written in the form

$$(\mathbf{y}, F\mathbf{y}) + (\mathbf{g}_3, P_3\mathbf{g}_3) + (\mathbf{y}, \nu_2\mathbf{g}_2 + \nu_3\mathbf{g}_3) + \sum_{i=1}^{3} (\mathbf{n}_i, \mathbf{g}_i) + (\mathbf{m}, \mathbf{y}) + h = \text{const} \tag{3.1}$$

$$\begin{aligned}
&(2F\mathbf{y} + \mathbf{m}, \mathbf{y} \times \mathbf{x} + \Lambda\mathbf{x} + p\mathbf{g}_3 \times J\mathbf{g}_3 + \mathbf{N}) + p\nu_2(\mathbf{g}_1, J\mathbf{g}_3) + \\
&+ \langle 2P_3\mathbf{g}_3 + \mathbf{n}_3, \mathbf{g}_3, \mathbf{x}\rangle + (\nu_2\mathbf{g}_2 + \nu_3\mathbf{g}_3, \Lambda\mathbf{x} + \mathbf{N}) + \\
&+ k_2(2P_3\mathbf{g}_3 + \mathbf{n}_3, \mathbf{g}_1) + \langle \mathbf{n}_2, \mathbf{g}_2, \mathbf{x}\rangle - k_3(\mathbf{n}_2, \mathbf{g}_1) - \langle \mathbf{n}_1, \mathbf{g}_1, \mathbf{x}\rangle - \\
&- k_2(\mathbf{n}_1, \mathbf{g}_3) + k_3(\mathbf{n}_1, \mathbf{g}_2) + (\nu_3 k_2 - \nu_2 k_3)(\mathbf{y}, \mathbf{g}_1) + (\mathbf{y}, F^\bullet \mathbf{y}) + \\
&+ (\mathbf{g}_3, P_3^\bullet \mathbf{g}_3) + (\mathbf{y}, \nu_2^\bullet \mathbf{g}_2 + \nu_3^\bullet \mathbf{g}_3) + \sum_{i=1}^{3} (\mathbf{n}_i^\bullet, \mathbf{g}_i) + (\mathbf{m}^\bullet, \mathbf{y}) + h^\bullet \equiv 0
\end{aligned} \tag{3.2}$$

Proposition 4. The operator F can be represented in the form

$$F = \psi_1 J^{-1} + \psi_2 E \tag{3.3}$$

Proof. Separating out terms with x^3 from identity (3.2), we obtain the identity

$$\langle FJ\mathbf{x}, J\mathbf{x}, \mathbf{x}\rangle \equiv 0$$

which, when the eigenvalues, A_i of the operator J are not equal to one another, is satisfied if and only if the operator F has the form (3.3).

Proposition 5. The operator P_3 can be represented in the form

$$P_3 = p(\psi_1 J - \psi_2 A_1 A_2 A_3 J^{-1}) \tag{3.4}$$

Proof. Grouping the terms with xg_3^2 in identity (3.2), we obtain the identity

$$p\langle FJ\mathbf{x}, \mathbf{g}_3, J\mathbf{g}_3\rangle + \langle P_3\mathbf{g}_3, \mathbf{g}_3, \mathbf{x}\rangle \equiv 0$$

which can be written in the equivalent form

$$pJF(\mathbf{g}_3 \times J\mathbf{g}_3) \equiv \mathbf{g}_3 \times P_3\mathbf{g}_3 \tag{3.5}$$

Assuming $\mathbf{g}_3 = \mathbf{e}_i$, we obtain $(P_3\mathbf{e}_i) \times \mathbf{e}_i = 0$, and then $\mathbf{e}_i$ $(i = 1, 2, 3)$ are the eigenvectors of the operator P_3. Taking Proposition 4 into account, we can rewrite identity (3.5) in the form

$$p[\psi_1 \mathbf{g}_3 \times J\mathbf{g}_3 + \psi_2 (J^2 \mathbf{g}_3) \times \mathbf{g}_3 + \psi_2 \operatorname{tr} J (\mathbf{g}_3 \times J\mathbf{g}_3)] \equiv \mathbf{g}_3 \times P_3 \mathbf{g}_3 \tag{3.6}$$

Here we have used the equality

$$D(\mathbf{a} \times \mathbf{b}) = (D^* \mathbf{b}) \times \mathbf{a} + \mathbf{b} \times D^* \mathbf{a} + \operatorname{tr} D(\mathbf{a} \times \mathbf{b})$$

Identity (3.6) is only satisfied if the operator P_3 has the form

$$P_3 = p(\psi_1 J - \psi_2 J^2 + \psi_2 J \operatorname{tr} J) + kE$$

Since

$$J^2 - J \operatorname{tr} J = A_1 A_2 A_3 J^{-1} - (A_1 A_2 + A_1 A_3 + A_2 A_3) E \tag{3.7}$$

we obtain representation (3.4) for the operator P_3. The term kE gives the function $k(t)$, which can be included in the term $h(t)$ in integral (2.1).

Proposition 6. The parameter $\mathbf{m}$ has the form

$$\mathbf{m} = -2F\mathbf{K} \tag{3.8}$$

Proposition 7. The operator P_3 is constant in the frame of reference E_3.

Proof. Taking into account the fact that $\mathbf{y} = J\mathbf{x} + \mathbf{K}$, we collect terms with g_3^2 in identity (3.2)

$$p\langle 2F\mathbf{K} + \mathbf{m}, \mathbf{g}_3, J\mathbf{g}_3 \rangle + (\mathbf{g}_3, P_3^{\bullet} \mathbf{g}_3) \equiv 0$$

It follows from Proposition 5 that $\mathbf{e}_i$ are the eigenvectors of the operator P_3. Substituting $\mathbf{g}_3 = \mathbf{e}_i$ into the last identity, we obtain $(\mathbf{e}_i, P_3^{\bullet} \mathbf{e}_i) = 0$ $(i = 1, 2, 3)$, whence Proposition 7 follows. The identity considered takes the form.

$$\langle 2F\mathbf{K} + \mathbf{m}, \mathbf{g}_3, J\mathbf{g}_3 \rangle \equiv 0$$

which is possible for pairwise different A_i, only if $2F\mathbf{k} + \mathbf{m} = 0$.

Proposition 8. The following representation holds

$$(\mathbf{y}, F\mathbf{y}) + (\mathbf{m}, \mathbf{y}) + h(t) = (J\mathbf{x}, FJ\mathbf{x}) + \text{const} \tag{3.9}$$

Proof. Taking Proposition 6 into account in identity (3.2) we separate out terms not containing $\mathbf{x}$ and $\mathbf{g}_i$

$$(\mathbf{K}, F^{\bullet} \mathbf{K}) + (\mathbf{m}^{\bullet}, \mathbf{K}) + h^{\bullet} \equiv 0$$

Hence

$$h^{\bullet} = -(\mathbf{K}, F^{\bullet} \mathbf{K}) + 2((F\mathbf{K})^{\bullet}, \mathbf{K}) = (\mathbf{K}, F\mathbf{K})^{\bullet} \text{ and } h = (\mathbf{K}, F\mathbf{K}) + \text{con}$$

The left-hand side of (3.9) can now be written in the form

$$(J\mathbf{x} + \mathbf{K}, FJ\mathbf{x} + F\mathbf{K}) - 2(F\mathbf{K}, J\mathbf{x} + \mathbf{K}) + (\mathbf{K}, F\mathbf{K}) + \text{const}$$

which is identical with the right-hand side of this equation.

Proposition 9. For integral (2.1) to exist it is necessary for the following conditions to be satisfied

$$\nu_2 = -2k_2 \psi_1 \tag{3.10}$$

$$k_2 \psi_2 = 0 \tag{3.11}$$

Proof. Separating out terms with $g_1 g_3$ in identity (3.2), we obtain

$$p\nu_2 (\mathbf{g}_1, J\mathbf{g}_3) + 2k_2 (P_3 \mathbf{g}_3, \mathbf{g}_1) \equiv 0$$

This identity is only satisfied if $p\nu_2 J + 2k_2 P_3 = 0$. Taking Proposition 5 into account, we obtain the condition

$$\nu_2 J + 2k_2(\psi_1 J - \psi_2 A_1 A_2 A_3 J^{-1}) = 0$$

Since all the eigenvalues of the operator J are different, this equality is only possible when conditions (3.10) and (3.11) are satisfied.

Proposition 10. When $k_2 \neq 0$, the operators F and P_3 have the form

$$F = \psi_1 J^{-1}, \quad P_3 = p\psi_1 J \tag{3.12}$$

The proof follows from Propositions 4, 5 and 9.

Proposition 11. For integral (2.1) to exist, the following conditions must be satisfied

$$\mathbf{n}_1 = 0 \tag{3.13}$$

$$k_3\nu_2 = k_2\nu_3 \tag{3.14}$$

Proof. Separating out terms with $g_1 x$ in identity (3.2), we obtain the identity

$$\langle \mathbf{n}_1, \mathbf{g}_1, \mathbf{x}\rangle + (k_2\nu_3 - k_3\nu_2)(J\mathbf{x}, \mathbf{g}_1) \equiv 0$$

which is only satisfied when conditions (3.13) and (3.14) are satisfied.

Proposition 12. For integral (2.1) to exist, the following conditions must be satisfied (for i = 2, 3)

$$\mathbf{n}_i = 0 \tag{3.15}$$

$$\nu_i \Lambda + \nu_i^{\bullet} J = 0 \tag{3.16}$$

Proof. Grouping terms with $g_2 x$ and $g_3 x$ in identity (3.2), we obtain the identities

$$\nu_i(\mathbf{g}_i, \Lambda\mathbf{x}) + \langle \mathbf{n}_i, \mathbf{g}_i, \mathbf{x}\rangle + \nu_i^{\bullet}(J\mathbf{x}, \mathbf{g}_i) \equiv 0, \quad i = 2,3$$

These identities are equivalent to the following

$$\nu_i \Lambda\mathbf{x} + \nu_i^{\bullet} J\mathbf{x} + \mathbf{x} \times \mathbf{n}_i \equiv 0$$

Since the operators Λ and J are symmetrical, the last identities are satisfied only when conditions (3.15) and (3.16) are satisfied.

Proposition 13. When $k_2 \neq 0$ the following representation holds

$$\nu_i = -2k_i\psi_1, \quad i = 2, 3$$

The proof follows from conditions (3.10) and (3.14).

Proposition 14. For integral (2.1) to exist, Condition 4 of Theorem 1 and the following condition must be satisfied

$$k_3 = c_1 k_2 \tag{3.17}$$

Proof. It follows from conditions (3.16) and Proposition 13 that $(k_3/k_2)^{\bullet} J = 0$, whence we obtain condition (3.17). When this condition is satisfied the two conditions (3.16) are equivalent to Condition 4 of Theorem 1, where

$$\eta = (k_2^2 \psi_1)^{-1} \tag{3.18}$$

Proposition 15. Condition (3.17) is satisfied if and only if the angular velocity of the orbital frame of reference has the form

$$\boldsymbol{\Omega} = k_2 \sin^{-1}\alpha \mathbf{e} \tag{3.19}$$

where $\mathbf{e}$ is the unit vector of an arbitrary direction, fixed in frames of reference E_0 and E_1. When this condition is satisfied, the centre of mass is shifted along the surface of a right circular cone, fixed in inertial space, with vertex at the centre of the force field and with an angle α between the generatrix and the axis of the cone, the direction of which is specified by the unit vector $\mathbf{e}$.

Proof. Suppose $k_3 = c_1 k_2$ and $k_2 \neq 0$. Then, taking formulae (1.1) into account, we obtain

$$(\boldsymbol{\Omega}/k_2)_0^{\cdot} = (\mathbf{g}_2 + c_1\mathbf{g}_3)_0^{\cdot} = \boldsymbol{\Omega}\times(\mathbf{g}_2 + c_1\mathbf{g}_3) = \boldsymbol{\Omega}\times(\boldsymbol{\Omega}/k_2) \equiv 0$$

Consequently, when condition (3.17) is satisfied we can write

$$\boldsymbol{\Omega} = k_2\mathbf{a}, \quad (\mathbf{a})_0^{\cdot} = 0 \tag{3.20}$$

Hence, the angular velocity Ω maintains its direction in E_0, but it then also has a constant direction in E_1. The orbital frame of reference, being displaced together with a centre of mass of the system, rotates around the fixed direction $\mathbf{a}$.

Multiplying Eq. (1.1) scalarly by $\mathbf{a}$ for $\mathbf{g}_3 = \boldsymbol{\gamma}$, we obtain $(\mathbf{a}, \boldsymbol{\gamma})^{\cdot} = 0$, and consequently, the angle between $\boldsymbol{\gamma}$ and $\mathbf{a}$ is constant and the trajectory of the centre of mass lies on a cone with vertex at the centre of the field.

We put $(\mathbf{a}, \boldsymbol{\gamma}) = a \cos\alpha = c$ and transform condition (3.17). By virtue of formulae (1.2) we obtain

$$k_2 = |\boldsymbol{\gamma}\times(\boldsymbol{\gamma})_0^{\cdot}| = k_2|\boldsymbol{\gamma}\times(\mathbf{a}\times\boldsymbol{\gamma})| = k_2|\mathbf{a} - c\boldsymbol{\gamma}| = k_2 a \sin\alpha$$

Hence it follows that the constants a and α are related by the condition

$$a \sin\alpha = 1 \tag{3.21}$$

which also enables us to represent the angular velocity Ω in the form (3.19). Further

$$k_3 = k_2^{-2}\langle\boldsymbol{\gamma},\ (\boldsymbol{\gamma})_0^{\cdot},\ (\boldsymbol{\gamma})_0^{\cdot\cdot}\rangle = k_2\langle\boldsymbol{\gamma},\ \mathbf{a}\times\boldsymbol{\gamma},\ \mathbf{a}\times(\mathbf{a}\times\boldsymbol{\gamma})\rangle = k_2(\mathbf{a} - c\boldsymbol{\gamma},\ c\mathbf{a}) = ck_2(a^2 - c^2) = ck_2a^2\sin^2\alpha$$

and, taking into account relation (3.21), we obtain $k_3 = c_1k_2$. Consequently, when condition (3.19) is satisfied the functions k_3 and k_2 are proportional, where $c_1 = c$.

Taking into account the propositions proved above, identity (3.2) can be written in the form

$$\begin{aligned} &2\psi_1(\mathbf{x},\ \Lambda\mathbf{x} + \mathbf{N}) + (\nu_2\mathbf{g}_2 + \nu_3\mathbf{g}_3,\ \mathbf{N}) + (J\mathbf{x},\ (\psi_1 J^{-1})^{\cdot} J\mathbf{x}) + 2(\mathbf{K},\ (\psi_1 J^{-1})^{\cdot} J\mathbf{x}) + \\ &+(\nu_2^{\cdot}\mathbf{g}_2 + \nu_3^{\cdot}\mathbf{g}_3,\ \mathbf{K}) - 2((\psi_1 J^{-1}\mathbf{K})^{\cdot},\ J\mathbf{x}) \equiv 0 \end{aligned} \tag{3.22}$$

Proposition 16. For integral (2.1) to exist, Condition 5 of Theorem 1 must be satisfied.

Proof. Separating out terms with x from identity (3.22), we obtain the condition

$$\psi_1\mathbf{N} + (\psi_1 J^{-1})^{\cdot} J\mathbf{K} - J(\psi_1 J^{-1}\mathbf{K})^{\cdot} = 0$$

which reduces to the form $\psi_1(\mathbf{N}-\mathbf{K}^{\cdot}) = 0$. Hence we obtain condition 5, since the case $\psi_1 = 0$ is eliminated when $k_2 \neq 0$. In fact, when $\psi_1 = 0$, it follows from Propositions 3 and 9–11 that $F = P_3 = \nu_2 = \nu_3 = 0$, and quadratic integral (3.1) does not exist.

Proposition 17. For integral (2.1) to exist, Condition 6 of Theorem 1 must be satisfied.

Proof. Separating out terms with $\mathbf{g}_2$ and $\mathbf{g}_3$ from identity (3.22), we obtain the condition

$$\nu_i\mathbf{N} + \nu_i^{\cdot}\mathbf{K} = 0, \quad i = 2, 3$$

which, by virtue of Proposition 16, can be written in the form $(\nu_i\mathbf{K})^{\cdot} = 0$. Taking Proposition 13 and formulae (3.17) and (3.18) into account, both these conditions can be written in the form $((k_2\eta)^{-1}\mathbf{K}^{\cdot}) = 0$, whence the necessity of condition 6 follows.

Proposition 18. For integral (2.1) to exist, Condition 3 of Theorem 1 must be satisfied.

Proof. Separating out terms with x^2 from identity (3.22), we obtain the condition

$$2\psi_1\Lambda + (\psi_1 J^{-1})^{\cdot} J^2 = 0$$

Substituting the expression for Λ from condition 4 here, we obtain $(\eta^{-1}J)^{\cdot} = 0$, whence the necessity of Condition 3 follows.

Proposition 19. Integral (2.1) exists only if the velocity of the centre of mass of the system satisfies Condition 2 of Theorem 1.

Proof. By Propositions 7 and 10

$$(p\psi_1 J)^{\cdot} = 0$$

Taking Condition 3 into account, where the parameter η is given by formula (3.18), we obtain the relation pk_2^{-2} = const, whence Condition 2 also follows.

Proposition 20. If integral (2.1) to exist, it can be written in the form (2.2) or (2.3).

Proof. Integral (2.1) can be reduced to the form (3.1), when, taking into account Proposition 8, 10–13 and representation (1.1) for the absolute angular velocity Ω of the orbital system, we obtain the following form of the integral

$$\psi_1[(\mathbf{x},\ J\mathbf{x}) + p(\gamma,\ J\gamma) - 2(J\mathbf{x}+\mathbf{K},\ \boldsymbol{\Omega})] = \text{const} \tag{3.23}$$

Taking into account the relation $\mathbf{x} = \mathbf{x}_{31} + \Omega$, we can write this integral in the form

$$\psi_1[(\mathbf{x}_{31},\ J\mathbf{x}_{31}) + p(\gamma,\ J\gamma) - (\boldsymbol{\Omega},\ J\boldsymbol{\Omega}) - 2(\mathbf{K},\ \boldsymbol{\Omega})] = \text{const} \tag{3.24}$$

We can change to the forms (2.2) and (2.3) if we take into account Conditions 3 and 6 of Theorem 1, expression (3.18) and the relation pk_2^{-2} = const, pointed out in the proof of Proposition 18.

It was shown above that Conditions 1–6 are necessary for non-trivial integral (2.1) to exist. These conditions are sufficient for the integral to exist, since the fundamental identity holds when they are satisfied. The sufficiency of these conditions also follows from the existence of integral (2.7) of system (2.4), (2.5), to which the initial system reduces (Proposition 32) when there is a non-trivial quadratic integral. Theorem 1 is completely proved.

4. PROOF OF THEOREM 2

We will now consider the case when $k_2 \equiv 0$.

Proposition 21. For a non-trivial quadratic integral to exist in the case when $k_2 \equiv 0$ it is necessary that the operators F and P_3 should have the form (3.3), (3.4), and the condition $P_3^{\cdot} = 0$ and condition (3.16) should be satisfied (for $i = 3$). In this case the integral can be written in the form (2.8) or in the form (2.17).

Proof. Proposition 1–9, 11 and 12 are also retained when $k_2 \equiv 0$. It follows from condition (3.10) that $\nu_2 = 0$, and condition (3.16) is identically satisfied for $i = 2$. Integral (2.1), taking Proposition 8 into account, can be written in the form (2.8), by putting $\nu = \nu_3$. If we use representation (3.3) and (3.4) for the operators F and P_3, the integral can be written in the form (2.17).

Identity (3.2), when the conditions mentioned in Proposition 21 are satisfied, reduces to the form

$$2(FJ\mathbf{x},\ \mathbf{K}\times\mathbf{x} + \Lambda\mathbf{x} + \mathbf{N}) + (J\mathbf{x},\ F^{\cdot}(J\mathbf{x}+2\mathbf{K}) + \mathbf{m}^{\cdot}) + (\nu\mathbf{N} + \nu^{\cdot}\mathbf{K},\ \gamma) \equiv 0 \tag{4.1}$$

Proposition 22. The operator F can be written in the form $F = F_1F_2$, where $\mathbf{e}_i$ (i = 1, 2, 3) are the eigenvectors of operators F_1 and F_2, the eigenvalues φ_{2i} of operator F_2 are constant, and the eigenvalues φ_{1i} of operators F_1 are specified by the equalities (here ($\lambda_{ij} = (\mathbf{e}_i, \Lambda\mathbf{e}_j)$))

$$\varphi_{1i} = \exp\left(-2\int_0^t A_i^{-1}(\xi)\lambda_{ii}(\xi)d\xi\right),\quad i = 1,\ 2,\ 3 \tag{4.2}$$

Proof. Separating out the terms with x^2 from identity (4.1), we obtain the identity

$$(J\mathbf{x},\ F^{\bullet}J\mathbf{x})+2(FJ\mathbf{x},\ \Lambda\mathbf{x})+2\langle FJ\mathbf{x},\ \mathbf{K},\ \mathbf{x}\rangle\equiv 0 \tag{4.3}$$

Assuming $\mathbf{x} = \mathbf{e}_i$ here, we obtain

$$\varphi_i^{\bullet}+2\varphi_i A_i^{-1}\lambda_{ii}=0 \tag{4.4}$$

whence it follows that $\varphi_i = \varphi_{1i}\varphi_{2i}$, where φ_{2i} = const, and φ_i is specified by formula (4.2).

We will represent the operator Λ in the form $\Lambda = \Lambda_1 + \Lambda_2$, where Λ_1 is the operator with eigenvectors and eigenvalues $\mathbf{e}_i$, λ_{ii} $(i = 1, 2, 3)$, then $(\mathbf{e}_i, \Lambda_2\mathbf{e}_j) = \lambda_{ij}(1 - \delta_{ij})$.

Proposition 23. When a non-trivial quadratic integral exists when $k_2 \equiv 0$, the following condition must be satisfied

$$\lambda_{ij}(A_i\varphi_i + A_j\varphi_j) = K^{(k)}(A_i\varphi_i - A_j\varphi_j)\delta_{ijk} \quad (i, j, k) \tag{4.5}$$

Proof. Taking Proposition 22 into account, identity (4.3) takes the form

$$(FJ\mathbf{x},\ \Lambda_2\mathbf{x}) + \langle FJ\mathbf{x},\ \mathbf{K},\ \mathbf{x}\rangle \equiv 0$$

This identity is equivalent to the system of conditions (4.5), which can be verified by decomposing, for example, $\mathbf{x}$ in the principal basis.

Proposition 24. For a non-trivial quadratic integral to exist when $k_2 \equiv 0$ the following condition must be satisfied

$$F_2(\mathbf{N} - \mathbf{K}^{\bullet}) = 0 \tag{4.6}$$

and if the operator F_2 is non-degenerate, we have

$$\mathbf{N} = \mathbf{K}^{\bullet} \tag{4.7}$$

Proof. Separating out terms that are linear in $\mathbf{x}$ from identity (4.1), and taking Proposition 6 into account, we obtain the condition

$$F\mathbf{N} + F^{\bullet}\mathbf{K} - (F\mathbf{K})^{\bullet} = 0$$

or

$$F(\mathbf{N} - \mathbf{K}^{\bullet}) = 0$$

Hence, we obtain condition (4.6), since, in accordance with Proposition 22, $F = F_1F_2$, where F_1 is a non-degenerate operator.

Proposition 25. For a quadratic integral to exist when $k_2 \equiv 0$ the following condition must be satisfied

$$\nu\mathbf{N} + \nu^{\bullet}\mathbf{K} = 0 \tag{4.8}$$

and if the operator F_2 is non degenerate, we have

$$(\nu\mathbf{K})^{\bullet} = 0 \tag{4.9}$$

Proof. By making terms with γ vanish in identity (4.1), we obtain condition (4.8). By virtue of Proposition 24 for the non-degenerate operator F_2 we obtain condition (4.9).

Proposition 26. The eigenvalues φ_i and p_i of the operators F and P_3 in the quadratic integral when $k_2 \equiv 0$ can be represented in the form (2.9), (2.10).

Proof. From (3.3) we obtain a representation for the eigenvalues φ_i of the operator F

$$\varphi_i = A_i^{-1}\psi_1 + \psi_2, \quad i = 1, 2, 3$$

The condition for this system of three equation to be consistent for ψ_1 and ψ_2 has the form

$$\varphi_1 a_1 + \varphi_2 a_2 + \varphi_3 a_3 = 0 \tag{4.10}$$

This condition is equivalent to the operator F being representable in the form (3.3)

Replacing the vector identity (3.5) by three scalar identities, we obtain that this identity is only satisfied provided.

$$\Delta p_i = p\varphi_i a_i, \quad i = 1, 2, 3 \tag{4.11}$$

We put

$$\sigma_i = p\varphi_i a_i$$

It follows from condition (4.10) that σ_i are related by condition (2.11). By virtue of Proposition 7, p_i = const and then σ_i = const. The system of linear equations (4.11) in p_1, p_2 and p_3 is consistent when condition (4.10) is satisfied; the set of its solutions can be written in the form

$$p_i = -\tfrac{1}{3}\Delta\sigma_i + \chi$$

Here χ = const (by virtue of the fact that p_i and σ_i are constant) and this term in p_i can be omitted, since it gives the constant term χg_3^2 in the quadratic integral.

Proposition 27. For a quadratic integral to exist when $k_2 \equiv 0$, condition (2.16) must be satisfied.

Proof. Substituting representation φ_i (2.9) into (4.4), we obtain condition (2.16).

Note that conditions (2.14) and (2.15) are obtained from conditions (4.5) and (4.6) using representation (2.9). Hence, the necessity of the conditions presented in Theorem 2 and the representability of the integral in the form (2.8) or (2.17) are proved. The sufficiency of the set of these conditions follows from the fact that the fundamental identity is satisfied when they are consistent.

5. PROOF OF THEOREM 3

Proposition 28. The integral of the projection of the kinetic moment, if it exists in the case $k_2 \equiv 0$, can be written in the form

$$\nu(J\mathbf{x} + \mathbf{K}, \boldsymbol{\gamma}) = \text{const} \tag{5.1}$$

For it to exist it is necessary and sufficient for condition (2.22) and the condition to be satisfied.

$$\mathbf{N} = -(\ln \nu)^{\boldsymbol{\cdot}}\mathbf{K} \tag{5.2}$$

When these conditions are satisfied the system can be reduced to the form (2.18).

Proof. Putting $\psi_1 = \psi_2 = 0$ in integral (2.17), we isolate integral (5.1). Conditions (2.12) and (2.13) when $\nu \neq 0$ can be written in the form (2.32) and (5.2), and the remaining conditions of Theorem 2 when $\sigma_i = 0$ (i = 1, 2, 3) are satisfied. The reducibility of the system to the form (2.18), where we take $J' = \nu J$, $\mathbf{K}' = \nu\mathbf{K}$, can be verified directly.

Proposition 29. For two integrals, quadratic in the components of the angular velocity, to exist, it is necessary to satisfy the second and third conditions of (2.22) and the conditions

$$\Lambda_2 = 0, \quad (\ln pa_i)^{\boldsymbol{\cdot}} = 2\lambda_{ii}A_i^{-1}, \quad i = 1, 2, 3 \tag{5.3}$$

Proof. By virtue of Theorem 2 these integrals can be written in the form (2.8). Taking representation (2.9) for the eigenvalues of the operator F into account, we obtain that independent sets of constants σ_i, satisfying condition (2.11), correspond to independent quadratic integrals. The solutions $\{\sigma_i^{(1)}\}$, $\{\sigma_i^{(2)}\}$ are the basis for the set of solutions of Eq. (2.11) with three variables, where

$$\sigma_i^{(1)} = \Delta A_{i0}, \quad \sigma_i^{(2)} = A_{i0}\Delta A_{i0}, \quad i = 1, 2, 3 \tag{5.4}$$

Here $A_{i0} = A_i(0)$.

By requiring that condition (2.14) must be satisfied both for $\sigma_i = \sigma_i^{(1)}$ and $\sigma_i = \sigma_i^{(2)}$, we obtain $\mathbf{K} = 0$ and $\lambda_{ij} = 0$ when $i \neq j$, i.e. $\Lambda_2 = 0$. From conditions (2.15) and (2.16) we now obtain the third condition of (2.22) and the second condition of (5.3).

Proposition 30. For three independent non-trivial quadratic integrals of the fundamental dynamic system to exist it is necessary for the last condition of (2.22) to be satisfied. The integrals can be written in the form (2.19)–(2.21).

Proof. According to Theorem 2 any non-trivial quadratic integral can be written in the form (2.8) and is determined by the function $\nu(t)$ and by the use of constant σ_i, related by condition (2.11). It then follows from the existence of three independent quadratic integrals that the integral of the projection of the kinetic moment (5.1) exists and, in accordance with Proposition 28, the necessity of satisfying the first condition of (2.22). From this condition and the second condition of (5.3) we obtain $p\nu^2 a_i = \text{const}$. Assuming, without loss of generality, that $p_0\nu_0^2 = 1$, we obtain the last condition of (2.22).

For the specified formula (5.4) of the set $\sigma_i^{(1)}$, from conditions (2.9), (2.10) and the last condition of (2.22) we obtain the following expressions for the eigenvalues of the operators F and P_3

$$\varphi_i = \nu^2 A_{i0}^{-1}, \quad p_i = A_{i0} - (A_{10} + A_{20} + A_{30})/3$$

which gives integral (2.19).

For the set $\sigma_i^{(2)}$ we similarly obtain

$$\varphi_i = \nu^2, \quad p_i = -A_{10}A_{20}A_{30}A_{i0}^{-1} + (A_{10}A_{20} + A_{10}A_{30} + A_{20}A_{30})/3$$

and the integral is written in the form (2.20).

Theorem 3 is completely proved, since we have proved the necessity of conditions (2.22) for the existence of three independent non-trivial quadratic integrals and the sufficiency of the set of these conditions for integrals (2.19)–(2.21) to exist. Any non-trivial quadratic integral is a linear combination of these integrals.

We will now describe the permissible laws of variation of the principal moments of inertia for the three quadratic integrals to exist.

Proposition 31. When three independent non-trivial quadratic integrals exist, the inertia operator of the system can be represented in the form

$$J = \theta(C + \eta E)^{-1} \tag{5.5}$$

where $\operatorname{tr} C = 0$.

Proof. Putting

$$\zeta_k = -A_i A_j p\nu^2 \quad (i \neq j \neq k, \ i \neq k) \tag{5.6}$$

the last condition of (2.22) can be written in the form

$$\Delta\zeta_i = a_{i0}, \quad i = 1, 2, 3$$

The general solution of this system has the form

$$\zeta_i = -\tfrac{1}{3}\Delta a_{i0} + \eta \tag{5.7}$$

We obtain from formulae (5.6)

$$A_k = -A_1 A_2 A_3 p\nu^2 \zeta_k^{-1}$$

which also gives representation (5.5), where $\theta = -A_1 A_2 A_3 p\nu^2$, and, by formula (5.7), the eigenvalues of the operator C are $-\tfrac{1}{3}\Delta a_{i0}$ $(i = 1, 2, 3)$.

6. SIMPLIFICATION OF THE DYNAMICAL SYSTEM IN THE PRESENCE OF INTEGRALS

We will indicate here the reductions to the autonomous system (2.4), (2.5) – when the conditions of Theorem 1 are satisfied, and to system (2.23) – when the conditions of Theorem 3 are satisfied, and we will prove Theorem 4, which gives the condition for reduction to system (2.26), integrable in quadratures.

Proposition 32. When the conditions of Theorem 1 are satisfied, the fundamental dynamical system reduces to the autonomous form (2.4), (2.5). Integral (2.2) of the initial system in this case can be written in the form (2.7).

Proof. When proving Proposition 19 we noted that Condition 2 of Theorem 1 is equivalent to the condition $pk_2^{-2} = \text{const}$ and can be written in the form $p = c_3k_2^2$, where $c_3 = c_2^{-2}$. Taking into account Conditions 2–6 of Theorem 1, Eq. (1.8) can be written in the form

$$J_0(\mathbf{x}^\bullet - (\ln k_2)^\bullet \mathbf{x}) = (J_0\mathbf{x} + k_2\mathbf{K}_0)\times\mathbf{x} + c_3k_2^2\mathbf{g}_3\times J_0\mathbf{g}_3$$

Changing to the variables $\mathbf{u}$ and τ, given by formulae (2.6), we obtain Eq. (2.4). Since Condition 1 can be represented in the form (3.17), Eqs (1.9) can be written in the form (2.5).

Changing to the variables (2.6) and taking into account representation (1.1) for the angular velocity Ω and the relations $k_3 = c_1k_2, p = c_3k_2^2$, integral (2.2) is converted to the form (2.7).

Proposition 33. When the conditions of Theorem 3 are satisfied, the fundamental dynamical system reduces to the form (2.23).

Proof. According to Proposition 28 the fundamental dynamical system reduces to the form (2.18), whence, taking into account the second condition of (2.22) and changing to the variable $\mathbf{w}$, given by the first formula of (2.24), we obtain

$$J_0\mathbf{w}^\bullet = \mathrm{v}^{-1}(J_0\mathbf{w})\times(J^{-1}J_0\mathbf{w}) + p\mathrm{v}\mathbf{g}_3\times J\mathbf{g}_3 \tag{6.1}$$

$$\mathbf{g}_3^\bullet = \mathrm{v}^{-1}\mathbf{g}_3\times(J^{-1}J_0\mathbf{w})$$

Note that the last condition of (2.22) is equivalent to the satisfaction of the identity

$$p\mathrm{v}^2J(J\mathbf{x}\times\mathbf{x}) \equiv J_0(J_0\mathbf{x}\times\mathbf{x}) \tag{6.2}$$

This assertion can be verified by writing identity (6.2) in coordinate form.

Using identity (6.2), the first equation of system (6.1) can be reduced to the form

$$J_0\mathbf{w}^\bullet = p^{-1}\mathrm{v}^{-3}B(J_0B\mathbf{w}\times B\mathbf{w}) + \mathrm{v}^{-1}B(\boldsymbol{\gamma}\times J_0\boldsymbol{\gamma}), \quad B = J_0J^{-1} \tag{6.3}$$

The following equation holds for the symmetrical operator B with eigenvalues β_i and for arbitrary vectors $\mathbf{b}_1$ and $\mathbf{b}_2$

$$B(B\mathbf{b}_1\times B\mathbf{b}_2) = \beta_1\beta_2\beta_3\mathbf{b}_1\times\mathbf{b}_2$$

by taking which into account Eq. (6.3) and the second equation of system (6.1) can be written in the form (2.23).

Theorem 4 is a consequence of Theorem 3 for the case of a similar change of the inertia operator. In fact, assuming $J = \lambda J_0$, formulae (2.24) can be written in the form

$$\mathbf{w} = \lambda\mathrm{v}\mathbf{x}, \quad d\tau = dt/(\lambda\mathrm{v}), \quad p(\lambda\mathrm{v})^2 = 1$$

whence $\lambda\mathrm{v} = p^{-1/2}$, and system (2.23) reduces to the form (2.26), and the integrals (2.19)–(2.21) can be written in the form (2.28)–(2.30). The condition $J = \lambda J_0$ when $p(\lambda\mathrm{v})^2 = 1$ can be written in the form of the first condition of (2.25), in which case the last condition of (2.22) is satisfied identically. The first condition of (2.22) takes the form of the last condition of (2.25).

REFERENCES

1. BELETSKII, V. V., *The Motion of an Artificial Satellite About a Centre of Mass.* Nauka, Moscow, 1965.
2. BRUN, F. de, Rotation kring en fix punkt. *Ofversigt. Kongl Svenska Vetenskaps-Akad. Vorhandl, Stookholm*, 1893, 7, 455–468.
3. KOOB, G. Sur le problème de la rotation d'un corps author d'un point fixé. *Bull. Soc. Mat.*, 1895, **23**, 210–216.
4. KHARLAMOVA, Ye. I., The motion of a rigid body around a fixed point in a central Newtonian force field. *Izv. Sib. Otd. Akad. Nauk SSSR*, 1959, **6**, 7–17.
5. GANTMAKHER, F. R. and LEVIN, L. M., The equations of motion of a rocket. *Prikl. Mat. Mekh.*, 1947, **11**, 3, 301–312.
6. OL'SHANSKII, V. Yu., The free motion of a complex mechanical system with quadratic integrals. Kosmich. Issled., 1996, **34**, 2, 145–149.
7. OL'SHANSKII, V. Yu., The reducibility of the equations of free motion of a complex mechanical system. *Prikl. Mat. Mekh.*, 1998, **62**, 5, 768–777.
8. OL'SHANSKII, V. Yu., Linear and quadratic integrals of a complex mechanical system. *Prikl. Mat. Mekh.*, 1996, **60**, 1, 37–46.

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